Stability properties and topology at infinity of f-minimal hypersurfaces

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Stability properties and topology at infinity of f–minimal hypersurfaces

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Abstract We study stability properties of f–minimal hypersurfaces isometrically immersed in weighted manifolds with non–negative Bakry–Émery Ricci curvature under volume growth conditions. Moreover, exploiting a weighted version of a finiteness result and the adaptation to this setting of Li–Tam theory, we investigate the topology at infinity of f–minimal hypersurfaces. On the way, we prove a new comparison result in weighted geometry and we provide a general weighted L^1–Sobolev inequality for hypersurfaces in Cartan–Hadamard weighted manifolds, satisfying suitable restrictions on the weight function.

Keywords f-minimal hypersurfaces · weighted manifolds · stability · finite index · topology at infinity

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1 Introduction

Many problems in geometric analysis lead to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. This turns out to be compatible with the metric structure of the manifold and the resulting spaces take the name of weighted manifolds, also known in the literature as manifolds with density. Weighted manifolds first arose in the study of diffusion processes on manifolds in works of D. Bakry and M. Émery, [1], and were intensively studied in recent years; see e.g. the seminal works of F. Morgan, [29], and G. Wei, W. Wylie, [46]. A weighted manifold is a triple $M_f^m = (M^m, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol}_M)$, where $(M^m, \langle\, ,\, \rangle)$ is a complete m–dimensional Riemannian manifold, $f \in C^\infty(M)$ and $d\mathrm{vol}_M$ denotes the canonical Riemannian volume form on M. The geometry of weighted manifolds is visible in the weighted metric structure, i.e., in the weighted measure of (intrinsinc) metric objects, and it is controlled by suitable concepts of curvature adapted to the density of the measure. In [1] (see also [25]), it was introduced an important generalization of Ricci curvature in this setting, known as Bakry–Émery Ricci tensor and defined as

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}(f).$$

Following M. Gromov, [17], if we consider an isometrically immersed orientable hypersurface Σ^m in the weighted manifold M_f, we can also define a generalization of the mean curvature vector field as

$$\mathbf{H}_f = \mathbf{H} + (\overline{\nabla} f)^\perp.$$

Here we have denoted by $\mathbf{H}$ the mean curvature vector field of the immersion, by $\overline{\nabla}$ the Levi–Civita connection on M, and by $(\cdot)^\perp$ the projection on the normal bundle of Σ.

It is a well–known fact that minimal hypersurfaces arise as critical points of the area functional. Since the weighted structure on M induces a weighted structure on Σ we can consider the variational problem for the weighted area functional

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f} d\mathrm{vol}_\Sigma.$$

From variational formulae, [2], one can see that Σ is f–minimal, namely a critical point of the weighted area functional, if and only if $\mathbf{H}_f$ vanishes identically.

Clearly, minimal hypersurfaces are a particular case of f–minimal hypersurfaces, corresponding to the case $f \equiv \mathit{const}$. Moreover, as we shall see more in details in Section 2, self–shrinkers of the mean curvature flow are important examples of f–minimal hypersurfaces in the Euclidean space with the Gaussian density $e^{-|x|^2/2}$. This, on one hand, gives a motivation to the study of f–minimal hypersurfaces and, on the other hand, strongly suggests to study self–shrinkers in the realm of weighted manifolds; this is the point of view adopted in [39], [37].

The research on f–minimal hypersurfaces has just started and it has been already approached by many authors; see e.g. [14], [19], [44], [8], [7], [26], [13], [43]. Much effort has been devoted to the study of the stability properties. As we will see later on, the stability properties of f–minimal hypersurfaces are taken into account by spectral properties of the following weighted Jacobi operator

$$L_f = -\Delta_f - \left(|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu)\right),$$

where $\mathbf{A}$ denotes the second fundamental form of the immersion, $\overline{\mathrm{Ric}}_f$ denotes the Bakry–Émery Ricci tensor of the ambient space, and $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the f–laplacian on Σ_f. Roughly speaking (for more details see Section 3 below) we say that an f–minimal hypersurface is L_f–stable if it minimizes the weighted area functional. The most up to date result, proved by X. Cheng, T. Mejia, and D. Zhou, [8], states that there exist no L_f–stable complete f–minimal hypersurfaces Σ immersed in a complete weighted manifold M_f with $\overline{\mathrm{Ric}}_f \geq k > 0$, provided $\mathrm{vol}_f(\Sigma) < +\infty$. Note that, by the equivalences obtained in [9], in the special case of self–shrinkers this conclusion was originally pointed out by T. Colding and W. Minicozzi in [10].

In the first part of the paper, we are able to generalize the result in [8], considering progressively weaker growth conditions on the intrinsic weighted volume growth of geodesic balls. Recall that, if $B_r(o)$ and $\partial B_r(o)$ denote respectively the metric ball and the metric sphere of Σ of radius $r > 0$ and centered at $o \in \Sigma$, we define

$$\mathrm{vol}_f(B_r(o)) = \int_{B_r(o)} e^{-f} d\mathrm{vol}_\Sigma, \qquad \mathrm{vol}_f(\partial B_r(o)) = \int_{\partial B_r(o)} e^{-f} d\mathrm{vol}_{m-1},$$

where $d\mathrm{vol}_{m-1}$ stands for the $(m-1)$–Hausdorff measure. We then prove the following theorem.

Theorem A *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$. Then there is no L_f–stable complete non–compact f–minimal hypersurface Σ immersed in M_f provided $\mathrm{vol}_f(B_r(o)) = O(\mathrm{e}^{\alpha r})$ as $r \to +\infty$, with $\alpha < 2\sqrt{k}$.*

Furthermore, in the instability case, exploiting the oscillatory behaviour of solutions of some ODEs that naturally arise in this setting, we investigate general geometric restrictions for the finiteness of the weighted index of the f–minimal hypersurface, that is, the maximum dimension of the linear space of compactly supported deformations that decrease the weighted area up to second order.

Theorem B *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$. Then there is no complete f–minimal hypersurface Σ immersed in M_f with $Ind_f(\Sigma) < +\infty$ provided one of the following conditions hold*

1. $\mathrm{vol}_f(\Sigma) = +\infty$ *and* $\mathrm{vol}_f(B_r(o)) \leq Cr^a$ *for any* $r \geq r_0$ *and some positive constants* C*,* r_0 *and* a*;*
2. $\mathrm{vol}_f(\partial B_r)^{-1} \notin L^1(+\infty)$ *and* $|A| \notin L^2(\Sigma, e^{-f}d\mathrm{vol}_\Sigma)$*.*

Note that this last research direction is significant also in the special case of self–shrinkers. We are not aware of any result in this direction up to now.

The second aim of this paper is to obtain information on the topology at infinity of f–minimal hypersurfaces immersed in suitable ambient spaces. We recall that, in the non–weighted setting, there is a well–known connection, developed by P. Li and L.–F. Tam and collaborators (see e.g. [21]), between the dimension of the space of L^2–harmonic forms, the number of non–parabolic ends, and the Morse index of the operator $-\Delta - a(x)$, where $-a(x)$ is the smallest eigenvalue of the Ricci tensor at x. Furthermore, following H. D. Cao, Y. Shen, S. Zhu, [18], and P. Li and J. Wang, [22], one shows that if the manifold supports a L^1–Sobolev inequality outside some compact set, then all ends are non–parabolic. According to D. Hoffman and J. Spruck, [20], this in particular applies to minimal submanifolds of Cartan–Hadamard manifolds.

In this order of ideas, by adapting the Li–Tam theory to the weighted setting and by means of a weighted version of an abstract finiteness result from [36], we are able to obtain the finiteness of the number of non–f–parabolic ends of a weighted manifold M_f, assuming the finiteness of the Morse index of the operator $-\Delta_f - a(x)$, where $-a(x)$ is now the smallest eigenvalue of Ric_f at x.

Using then the technique adopted in [27], [20], we are able to guarantee the validity of a weighted L^1–Sobolev inequality outside some compact set on f–minimal hypersurfaces with finite weighted index, under suitable assumptions on f and on the curvature of the ambient weighted manifold. On the way we prove a comparison theorem in weighted geometry assuming an upper bound on the sectional curvature. An adaptation to the weighted setting of the results in [18], [22] finally provides the following topological result.

Theorem C *Let Σ^m be a complete f–minimal hypersurface isometrically immersed with $Ind_f(\Sigma) < +\infty$ in a complete weighted manifold M_f^{m+1} with $\overline{Sect} \leq 0$ and $\overline{\mathrm{Ric}}_f \geq k \geq 0$. Suppose furthermore that $f^* = \sup_\Sigma f < +\infty$ and $|\overline{\nabla} f| \in L^m(\Sigma_f)$. Then Σ has finitely many ends.*

As a consequence, adapting ideas in [24], we are able to obtain the following result, in which we replace the finiteness of the weighted index with the finiteness of the weighted total curvature of the f–minimal hypersurface.

Corollary D *Let Σ^m be a complete f–minimal hypersurface isometrically immersed in a complete weighted manifold M_f^{m+1} with $\overline{Sect} \leq 0$ and $\overline{\mathrm{Ric}}_f \geq k \geq 0$. Assume that $|\mathbf{A}| \in L^m(\Sigma_f)$. Suppose furthermore that $f \leq f^* < +\infty$ and $|\overline{\nabla} f| \in L^m(\Sigma_f)$. Then Σ has finitely many ends.*

The paper is organized as follows. In Section 2 we introduce some notation and provide some examples of f–minimal hypersurfaces. Section 3 is devoted to the study of stability properties of f–minimal hypersurfaces. Namely we analyze geometric conditions for the instability and infiniteness of the weighted index of these objects. In Section 4 we present a weighted version of an abstract finiteness result, recently obtained in [35], and state the adapted Li–Tam theory in the weighted setting. In Section 5 we prove a new comparison result in weighted geometry. In Section 6 a proof of the weighted L^1–Sobolev inequality for hypersurfaces in Cartan–Hadamard manifolds is provided. We end the paper with Section 7, where we finally prove the topological Theorem 7 and Corollary 2.

2 Definitions and some examples

Let $M_f^{m+1} = (M^{m+1}, \langle \cdot\,,\,\cdot \rangle, \mathrm{e}^{-f}d\mathrm{vol}_M)$ be a weighted manifold and let Σ^m be an isometrically immersed orientable hypersurface. We will denote by $\mathbf{A}$ the second fundamental form of the immersion $x : \Sigma^m \to M_f^{m+1}$, that is

$$\mathbf{A}(X, Y) = (\overline{\nabla}_X Y)^\perp,$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on M and $(\cdot)^\perp$ denotes the projection on the normal bundle of Σ. Denote by $\mathbf{H} = \mathrm{tr}_\Sigma \mathbf{A}$ the mean curvature vector field of the immersion. We define the *f–mean curvature vector field* of Σ as

$$\mathbf{H}_f = \mathbf{H} + (\overline{\nabla} f)^\perp.$$

Hence, denoting by ν be the unit normal we define the *f–mean curvature* H_f of Σ by $\mathbf{H}_f := H_f \nu$

Definition 1 Let $x : \Sigma^m \to M_f^{m+1}$ be a connected isometrically immersed hypersurface. We say that Σ is *f–minimal* if $\mathbf{H}_f \equiv 0$.

Remark 1 Note that, when f is constant $\mathbf{H}_f = \mathbf{H}$ and we recover the usual definition of a minimal hypersurface.

Example 1 (Self–shrinkers) Let Σ^m be a complete m–dimensional Riemannian manifold without boundary smoothly immersed by $x_0 : \Sigma^m \to \mathbb{R}^{m+1}$ as an hypersurface in the Euclidean space $\mathbb{R}^{m+1}$. We say that $\Sigma_0 = x_0(\Sigma^m)$ is moved along its mean curvature vector if there is a 1–parameter family of smooth immersions $x : \Sigma^m \times [t_0, T) \to \mathbb{R}^{m+1}$, with corresponding hypersurfaces $\Sigma_t = x(\cdot\,,\,t)(\Sigma^m)$, such that it satisfies the following mean curvature flow initial value problem

$$\begin{cases} \frac{\partial}{\partial t} x(p, t) = \mathbf{H}(p, t) \\ x(\cdot\,,\,t_0) = x_0, \end{cases} \tag{1}$$

for any $p \in \Sigma^m$, $t \in [t_0, T)$. Here $\mathbf{H}(p, t)$ is the mean curvature vector field of the hypersurface M_t at $x(p, t)$. The short time existence and uniqueness of a solution of (1) was investigated in classical works on quasilinear parabolic equations.

A MCF $\{\Sigma_t\}_{t<0}$ is called a self–shrinking solution if it satifies

$$\Sigma_t = \sqrt{-2t}\Sigma_{-\frac{1}{2}}$$

for all $t < 0$. For an overview on the role that such solutions play in the study of MCF see e.g. the introduction in [10]. An hypersurface is said to be a self–shrinker if it is the time $t = -\frac{1}{2}$ slice of a self–shrinking solution. Equivalently, by a self shrinker (based at $0 \in \mathbb{R}^{m+1}$) we mean a connected, isometrically immersed hypersurface $x : \Sigma^m \to \mathbb{R}^{m+1}$ whose mean curvature vector field satisfies the equation

$$x^{\perp} = -\mathbf{H}. \tag{2}$$

Let $f = \frac{|x|^2}{2}$ and consider the Gaussian space $\mathbb{R}^{m+1}_f$, which is the Euclidean space endowed with the canonical metric and the measure $\mathrm{e}^{-|x|^2/2}d\mathrm{vol}_{\mathbb{R}^{m+1}}$. A simple computation shows that

$$\overline{\nabla} f = x,$$

We hence obtain that f–minimal hypersurfaces Σ in the Gaussian space $\mathbb{R}^{m+1}_f$ satisfy

$$\mathbf{H} + x^{\perp} = 0,$$

and thus are exactly the self–shrinkers of mean curvature flow.

Example 2 (Slices of warped products of the form $P \times_{e^{-f}} \mathbb{R}$) Let $M^{m+1} = P^m \times_{e^{-f}} \mathbb{R}$, where P is an m–dimensional Riemannian manifold, $f : P \to \mathbb{R}_+$ is a smooth function and the product manifold $P \times \mathbb{R}$ is endowed with the Riemannian metric

$$\langle \cdot\,,\,\cdot \rangle = \pi_P^*(\langle \cdot\,,\,\cdot \rangle_P) + e^{-2f(\pi_P)}\pi_{\mathbb{R}}^*(dt \otimes dt).$$

Here $\pi_{\mathbb{R}}$ and π_P denote the projections onto the corresponding factors and $\langle \cdot\,,\,\cdot \rangle_P$ is the Riemannian metric on P^m. It is a well–known fact (see for instance [33]) that the distribution on the space orthogonal to $T = \partial/\partial t$ provides a foliation of M by means of totally geodesic (hence minimal) leaves $P_t = P \times \{t\}$, $t \in \mathbb{R}$. Moreover, since the function f only depends on P, it follows that the unit normal $\nu_t = T$, is everywhere orthogonal to $\overline{\nabla} f$. Hence the slices P_t, $t \in \mathbb{R}$, represent a distinguished family of f–minimal hypersurfaces in M.

3 Stability properties

It is a well–known fact that minimal hypersurfaces arise from a variational problem. Indeed, they are critical points of the area functional

$$\mathrm{vol}(\Sigma) = \int_{\Sigma} d\mathrm{vol}_{\Sigma}.$$

More precisely, letting x_t , $t \in (-\varepsilon, \varepsilon)$, $x_0 = x$, be a smooth compactly supported variation of immersions and denoting by V the associated variational vector field along x one gets that

$$\frac{d}{dt}\mathrm{vol}(x_t(\Sigma))\Big|_{t=0} = -\int_\Sigma \langle \mathbf{H}, V\rangle d\mathrm{vol}_\Sigma.$$

A similar characterization can be given also for f–minimal hypersurfaces, (see e.g [44], [8]). Indeed, defining the weighted area functional of $\Sigma^m \to M_f^{m+1}$ by

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f} d\mathrm{vol}_\Sigma$$

we have that

$$\frac{d}{dt}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} = -\int_\Sigma \langle \mathbf{H}_f, V\rangle e^{-f} d\mathrm{vol}_\Sigma.$$

We can now give the following

Definition 2 Let x_t , $t \in (-\varepsilon, \varepsilon)$, $x_0 = x$, be a smooth compactly supported variation of immersions. We say that a f–minimal hypersurface Σ is L_f–stable if

$$\frac{d^2}{dt^2}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} \geq 0.$$

Denote by V the variational vector field along x associated to the variation and let $V = u\nu$, $u \in C_c^\infty$. By a direct computation one can prove the following second variation formula for the weighted area, [2],

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} &= \int_\Sigma (|\nabla u|^2 - u^2(|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu)))\mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma \\ &= \int_\Sigma u\, L_f u\, \mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma,\end{aligned}$$

where $\overline{\mathrm{Ric}}_f$ is the Bakry–Émery Ricci tensor of M_f^{m+1}, and the operator L_f is defined by

$$L_f u = -\Delta_f u - (|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu))u.$$

Some steps into the study of non–existence results for L_f stable f–minimal hypersurfaces were moved in [14], [8], [26].

Proposition 1 ([14], Corollary 1.4, [26], Theorem 1) *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k$ and let Σ be a compact f–minimal hypersurface immersed in M_f .*

(a) If $k > 0$ then Σ cannot be L_f–stable;
(b) If $k = 0$ and Σ is L_f–stable, then it has to be totally geodesic and $\overline{\mathrm{Ric}}_f(\nu,\nu) = 0$.

Proposition 2 ([8], Theorem 5) *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$. Then there exists no L_f–stable complete f–minimal hypersurface Σ immersed in M_f without boundary and with $\mathrm{vol}_f(\Sigma) < +\infty$.*

Remark 2 When Σ is a complete self–shrinker with $\mathrm{vol}_f(\Sigma) < +\infty$, the conclusion in Proposition 2 was originally pointed out by T. Colding an W. Minicozzi in [10]. This follows by the equivalences obtained in [9]. Note also that it was conjectured by H. D. Cao that the weighted volume of complete self–shrinkers is always finite. On the other hand there is still no real evidence for this conjecture; see the very recent [37] where some steps in this direction are made.

Following classical terminology in linear potential theory recall that a weighted manifold $M_f = \left(M, \langle\cdot\,,\,\cdot\rangle\,, e^{-f}d\mathrm{vol}_M\right)$ is said to be f–parabolic if every solution of $\Delta_f u \geq 0$ satisfying $u^* = \sup_M u < +\infty$ must be identically constant.

Moreover, the positivity of a Schrödinger operator can be formulated in term of the existence of positive solutions of the associated linear equation. Indeed the following equivalence holds, that is a weighted version of a classical result by D. Fischer–Colbrie and R. Schoen, [15], and W. F. Moss and J. Piepenbrink, [31].

Proposition 3 ([45]) *Let M_f be a weighted manifold, and $\Omega \subset M$ be a domain in M and let $L = -\Delta_f + q\,(x)$, $q\,(x) \in L^{\infty}_{loc}\,(\Omega)$. Denote by $\lambda_1^L(\Omega)$ the bottom of the spectrum of L on Ω. The following facts are equivalent.*

(i) There exist $\omega \in C^{1,\alpha}_{loc}\,(\Omega)$, $\omega > 0$, weak solution of

$$\Delta_f \omega - q\,(x)\,\omega = 0 \quad on \quad \Omega.$$

(ii) There exist $\varphi \in W^{1,2}_{loc}\,(\Omega)$, $\varphi > 0$, weak solution of

$$\Delta_f \varphi - q\,(x)\,\varphi \leq 0 \quad on \quad \Omega.$$

(iii) $\lambda_1^L\,(\Omega) \geq 0$.

The proof of the previous proposition is straightforward once one looks at Lemma 3.10 in [36] and observes that $L = -\Delta_f + q\,(x)$ is unitarily equivalent to the Schrödinger operator

$$S = -\Delta + \left[(1/4\,\langle\nabla f, \nabla f\rangle - 1/2\Delta f) + q\,(x)\right] = -\Delta + (p\,(x) + q\,(x))$$

under the multiplication map $T = M_{e^{-f/2}} : L^2\left(M, e^{-f}d\mathrm{vol}_M\right) \to L^2\left(M, d\mathrm{vol}_M\right)$.

Example 3 (L_f–stable f–minimal hypersurfaces in warped products) Let $M^{m+1} = P^m \times_{e^{-f}} \mathbb{R}$, as in Example 2, and let Σ^m be an f–minimal hypersurface isometrically immersed in M^{m+1}. Setting $Y = e^{-f}T$ it is not hard to prove that the function $u = \langle Y, \nu\rangle$ satisfies

$$\Delta_f u + (|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu))u = 0.$$

Hence, using the previous proposition, we can see that every f–minimal hypersurface Σ isometrically immersed in M satisfying $0 < u$ is L_f–stable.

We can now obtain the following generalization of Proposition 1 and Proposition 2. Note that point (b) was already obtained in the very recent Theorem 7.3 in [13].

Proposition 4 *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k \geq 0$ and let Σ be a f–parabolic, complete, f–minimal hypersurface immersed in M_f.*

(a) If $k > 0$ then Σ cannot be L_f–stable.
(b) if $k = 0$ and Σ is L_f–stable, then it has to be totally geodesic and $\overline{\mathrm{Ric}}_f(\nu, \nu) = 0$.

Proof Assume that Σ is a L_f–stable complete f–minimal hypersurface immersed in M_f which is f–parabolic. Since Σ is L_f–stable, it follows by Proposition 3 that there exists a nonconstant function $u \in W^{1,2}_{loc}(\Sigma_f)$, $u > 0$, weak solution of

$$\Delta_f u + (|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu))u = 0.$$

Since $\overline{\mathrm{Ric}}_f$ is bounded below by a positive constant k, this also implies that u is a weak solution of

$$\Delta_f u \leq -(k + |\mathbf{A}|^2)u \leq 0.$$

Hence u is a f–superharmonic function bounded from below and, since Σ is f–parabolic, it must be constant. In particular,

$$|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu) = 0.$$

and the conclusion follows immediately.

Remark 3 It can be shown that a sufficient condition for Σ to be f–parabolic is that it is geodesically complete and

$$\mathrm{vol}_f(\partial B_r)^{-1} \notin L^1(+\infty). \tag{3}$$

This fact can be easily established adapting to the diffusion operator Δ_f standard proofs for the Laplace–Beltrami operator; see [16], [38].
Moreover, note that it is not difficult to prove that f–parabolicity is also guaranteed if we assume the stronger condition

$$\mathrm{vol}_f(B_r) = O(r^2), \quad \text{as } r \to +\infty. \tag{4}$$

The previous formula shows also that f–parabolicity holds if the manifold Σ has finite f–volume. Hence, in particular, the conclusion in Proposition 4 can be obtained if we either assume $\mathrm{vol}_f(\Sigma) < +\infty$ or $\mathrm{vol}_f(B_r(o)) = O(r^2)$ as $r \to +\infty$.

In the following result we show that one can do even better, assuming a weaker growth condition on the weighted volume of geodesic balls.

Theorem 1 *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$. Then there is no L_f–stable complete non–compact f–minimal hypersurface Σ immersed in M_f provided $\mathrm{vol}_f(B_r(o)) = O(\mathrm{e}^{\alpha r})$ as $r \to +\infty$, with $\alpha < 2\sqrt{k}$.*

Proof Define the weighted volume entropy of $(\Sigma, \langle \cdot\,,\,\cdot \rangle_\Sigma\,, \mathrm{e}^{-f}d\mathrm{vol}_\Sigma)$ as

$$h_f(\Sigma) := \limsup_{r\to+\infty} \frac{\log \mathrm{vol}_f(B_r(o))}{r}.$$

As observed in [4], the following inequality holds true in general for the bottom of the spectrum of the f–Laplacian λ_1^f:

$$\lambda_1^f(\Sigma) \leq \frac{1}{4} h_f^2(\Sigma).$$

Hence, in particular, if we assume that $\mathrm{vol}_f(B_r(o)) = O(\mathrm{e}^{\alpha r})$ as $r \to +\infty$ we obtain

$$\lambda_1^f(\Sigma) \leq \frac{\alpha^2}{4}.$$

Now assume by contradiction that Σ is L_f–stable. Then

$$\begin{aligned}\frac{\alpha^2}{4} \geq \lambda_1^f(\Sigma) = & \inf_{0\neq u\in C_c^\infty(\Sigma)} \frac{\int_\Sigma |\nabla u|^2 \mathrm{e}^{-f} dvol_\Sigma}{\int_\Sigma u^2 \mathrm{e}^{-f} d\mathrm{vol}_\Sigma} \\ \geq & \frac{\int_\Sigma (|A|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu)) u^2 \mathrm{e}^{-f} d\mathrm{vol}_\Sigma}{\int_\Sigma u^2 \mathrm{e}^{-f} d\mathrm{vol}_\Sigma} \\ \geq & k,\end{aligned}$$

for any $u \in C_c^\infty(\Sigma)$, contradicting the assumption on α.

In order to study L_f–unstable f–minimal hypersurfaces we introduce the f–index of Σ as the generalized Morse index of L_f on Σ. Namely, let $x : \Sigma^m \to M_f^{m+1}$ be an isometrically immersed complete orientable f–minimal hypersurface. Given a bounded domain $\Omega \subset \Sigma$ we define

$$Ind^{L_f}(\Omega) = \#\{\text{negative eigenvalues of } L_f \text{ on } \mathcal{C}_0^\infty(\Omega)\}.$$

The f*–index* of Σ is then defined as

$$Ind_f(\Sigma) := Ind^{L_f}(\Sigma) = \sup_{\Omega \subset\subset \Sigma} Ind^{L_f}(\Omega).$$

Geometrically, the f–index of Σ can be described as the maximum dimension of the linear space of compactly supported deformations that decrease the weighted area up to second order.

The following result, due to B. Devyver, [11], permits to interpret the finiteness of the Morse index of a Schrödinger operator in terms of the existence of a positive solution of the associated linear equation outside a compact set (also in the weighted setting).

Proposition 5 *Let Σ_f be a complete weighted manifold, and let $L = -\Delta_f - q(x)$, $q(x) \in L^\infty_{loc}(\Sigma)$. The following facts are equivalent*

(i) L has finite Morse index.

(ii) *There exists a positive smooth function* $\varphi \in W^{1,2}_{loc}$ *which satisfies* $L\varphi = 0$ *outside a compact set.*

(iii) $\lambda_1^L(\Sigma \setminus \Omega) \geq 0$, *for some* $\Omega \subset\subset \Sigma$.

Let $v(t) = \mathrm{vol}_f(\partial B_t(o))$, where $\partial B_t(o)$ are the geodesic spheres of radius t in Σ. Note that by the co-area formula we have that

$$\mathrm{vol}_f(B_r(o)) = \int_0^r v(t)dt. \tag{5}$$

We obtain the following

Proposition 6 *Let* Σ_f *be a complete noncompact weighted manifold with* $\mathrm{vol}_f(\Sigma) = +\infty$ *and let* Ω *be an arbitrary compact subset of* Σ. *Then*

1. *If* $\mathrm{vol}_f(B_r(o)) \leq Cr^a$ *for any* $r \geq r_0$ *and some positive constants* C, r_0 *and* a, *then* $\lambda_1^f(\Sigma \backslash \Omega) = 0$.
2. *If* $\mathrm{vol}_f(B_r(o)) \leq C\mathrm{e}^{\alpha r}$ *for any* $r \geq 0$ *and some positive constants* C *and* α, *then* $\lambda_1^f(\Sigma \backslash \Omega) \leq \frac{\alpha^2}{4}$.

Proof Since Ω is compact we can find a constant T_0 such that $\Omega \subset B_{T_0}(o)$. We reason now as in [12], [3], and exploit the oscillatory behaviour under our assumptions of solutions of the ODE

$$\begin{cases} (v(t)x'(t))' + \lambda v(t)x(t) = 0, \quad \text{a.e. on} \quad (T_0, +\infty), \\ x(T_0) = x_0, \end{cases} \tag{6}$$

where $v(t)$ is a positive continuous function on $[T_0, +\infty)$ and λ is a positive constant. Choosing $v(t) = \mathrm{vol}_f(\partial B_t(o))$ it then follows from Theorem 2.1 in [12] that equation (6) is oscillatory provided Σ has infinite f–volume and either the assumption in (1), or $\lambda > \frac{\alpha^2}{4}$ and the assumption in (2), hold true. Now the proof proceeds with slight modifications as in Theorem 3.1 in [12], but we report it here for the sake of completeness. Let us first assume that $\mathrm{vol}_f(B_r(o)) \leq Cr^a$ for any $r \geq r_0$ and some positive constants C, r_0, a. Then for any $\lambda > 0$ there exists some nontrivial oscillatory solution $x_\lambda(t)$ of (6) a.e. on $[T_0, +\infty)$, i.e., there exist T_1^λ and T_2^λ in $[T_0, +\infty)$ such that $T_1^\lambda < T_2^\lambda$, $x_\lambda(T_1^\lambda) = x_\lambda(T_2^\lambda) = 0$, and $x_\lambda(t) \neq 0$ for any $t \in (T_1^\lambda, T_2^\lambda)$. Let $\varphi_\lambda(x) = x_\lambda(r(x))$ and $\Omega_\lambda = B_{T_2^\lambda}(o) \setminus B_{T_1^\lambda}(o)$. It follows that

$$\begin{aligned} 0 \leq \lambda_1^f(\Sigma \setminus \Omega) &\leq \lambda_1^f(\Omega_\lambda) \\ &\leq \frac{\int_{\Omega_\lambda} |\nabla\varphi_\lambda|^2 e^{-f} d\mathrm{vol}_\Sigma}{\int_{\Omega_\lambda} |\varphi_\lambda|^2 e^{-f} d\mathrm{vol}_\Sigma} \\ &= \frac{\int_{T_1^\lambda}^{T_2^\lambda} (x'_\lambda(r))^2 v(r)dr}{\int_{T_1^\lambda}^{T_2^\lambda} (x_\lambda(r))^2 v(r)dr} \\ &= -\frac{\int_{T_1^\lambda}^{T_2^\lambda} (v(r)x'_\lambda(r))' x_\lambda(r)dr}{\int_{T_1^\lambda}^{T_2^\lambda} (x_\lambda(r))^2 v(r)dr} \\ &= \lambda. \end{aligned}$$

Since λ is an arbitrary positive constant, we obtain that $\lambda_1^f(\Sigma \setminus \Omega) = 0$.

On the other hand, suppose that the assumption in (2) is satisfied. Then, for any $\lambda > \frac{\alpha^2}{4}$ there exists again a nontrivial oscillatory solution $x_\lambda(t)$ of (6) on $[T_0, +\infty)$. Proceeding as above, we get that $\lambda_1^f(\Sigma \setminus \Omega) \leq \lambda$. The conclusion is thus straightforward since λ is an arbitrary positive constant larger than $\frac{\alpha^2}{4}$.

Adapting arguments in [3] we obtain also the following

Proposition 7 *Let Σ_f be a complete non–compact weighted manifold and let L be the Schrödinger operator defined by*

$$Lu = -\Delta_f u - q(x)u,$$

where $q(x)$ is a continuous nonnegative function on Σ. Assume that

(i) $\mathrm{vol}_f(\partial B_r(o))^{-1} \notin L^1(+\infty)$*;*
(ii) $q \notin L^1(\Sigma, e^{-f} d\mathrm{vol}_\Sigma)$*.*

Then, for an arbitrary compact subset $\Omega \subset \Sigma$ we have that the bottom of the spectrum of L on $\Sigma \setminus \Omega$ satisfies $\lambda_1^L(\Sigma \setminus \Omega) < 0$.

Proof Since Ω is compact we can find a constant T_0 such that $\Omega \subset B_{T_0}(o)$. By Corollary 2.4 in [3] we have that under our assumptions any solution $x(t)$ of

$$\begin{cases} (v(t)x'(t))' + Q(t)v(t)x(t) = 0, \quad \text{a.e. on} \quad (T_0, +\infty), \\ x(T_0) = x_0 \end{cases} \tag{7}$$

where $Q(t) = \frac{1}{v(t)} \int_{\partial B_t(o)} q e^{-f}$, is oscillatory. Choose, as above, $v(t) = \mathrm{vol}_f(\partial B_t(o))$. Then there exists some nontrivial oscillatory solution $x_Q(t)$ of (7) a.e. on $[T_0, +\infty)$, i.e., there exist T_1^Q and T_2^Q in $[T_0, +\infty)$ such that $T_1^Q < T_2^Q$ and $x_Q(T_1^Q) = x_Q(T_2^Q) = 0$, and $x_Q(t) \neq 0$ for any $t \in (T_1^Q, T_2^Q)$. Let $\varphi_Q(x) = x_Q(r(x))$ and $\Omega_Q = B_{T_2^Q}(o) \setminus B_{T_1^Q}(o)$. Using the co–area formula (5) we get

$$\begin{aligned} \int_{\Omega_Q} (|\nabla \varphi_Q|^2 - q\varphi_Q^2) e^{-f} d\mathrm{vol}_\Sigma &= \int_{\Omega_Q} (x_Q'(r)^2 - q x_Q(r)^2) e^{-f} d\mathrm{vol}_\Sigma \\ &= \int_{T_1^Q}^{T_2^Q} (x_Q'(r)^2 v(r) - x_Q(r)^2 Q(r) v(r)) dr \\ &= -\int_{T_1^Q}^{T_2^Q} x_Q(r)((v(r)x_Q'(r))' + Q(r)v(r)x_Q(r)) dr \\ &= 0. \end{aligned}$$

The conclusion follows now by strict domain monotonicity.

The previous results, applied in the setting of f–minimal hypersurfaces allow us to obtain the following

Theorem 2 *Let M_f be a complete weighted manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$. Then there is no complete f–minimal hypersurface Σ immersed in M_f with $Ind_f(\Sigma) < +\infty$ provided one of the following conditions holds*

1. $\mathrm{vol}_f(\Sigma) = +\infty$ *and* $\mathrm{vol}_f(B_r(o)) \leq Cr^a$ *for any* $r \geq r_0$ *and some positive constants* C, r_0 *and* a*;*
2. $\mathrm{vol}_f(\partial B_r)^{-1} \notin L^1(+\infty)$ *and* $|A| \notin L^2(\Sigma, e^{-f} d\mathrm{vol}_\Sigma)$.

Remark 4 Observe that if $\mathrm{vol}_f(\Sigma) < +\infty$ then $\mathrm{vol}_f(\partial B_r)^{-1} \notin L^1(+\infty)$. Indeed,by the Cauchy–Schwartz inequality, we have that for all $R > 0$ and $r > R$,

$$\int_R^r \frac{ds}{\mathrm{vol}_f(\partial B_s)} \int_R^r \mathrm{vol}_f(\partial B_s) ds \geq (r-R)^2.$$

Taking now the limit as $r \to \infty$ the conclusion follows. Hence, the case of finite f–volume is taken into account in part (2) of the theorem.

Proof (of Theorem 2) Assume that $\mathrm{Ind}_f(\Sigma) < +\infty$, $\mathrm{vol}_f(\Sigma) = +\infty$ and

$$\mathrm{vol}_f(B_r(o)) \leq Cr^a$$

for any $r \geq r_0$ and some positive constants C, r_0 and a. Then, for all $r \geq r_0$,

$$0 \geq \inf_{\Sigma \setminus B_r(o)} (|A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu)).$$

This gives a contradiction in case $\overline{\mathrm{Ric}}_f \geq k > 0$.

To get the proof of the remaining case we only have to observe that if $|A| \notin L^2(\Sigma, e^{-f} d\mathrm{vol}_\Sigma)$ then $q = |A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu) \notin L^1(e^{-f} d\mathrm{vol}_\Sigma)$. Hence under the assumption $\mathrm{vol}_f(\partial B_r(o))^{-1} \notin L^1(+\infty)$ we can apply Proposition 7 to conclude the proof.

4 Finiteness results and weigthed Li–Tam theory

Finiteness results for L^2 harmonic sections have been extensively investigated by many authors under different assumptions. With respect to this, we quote [22], [23], [6], [35], [36].

The abstract finiteness result we are going to present is an adaptation to the weighted setting of Theorem 1.1 in [35]; see also [36].

Theorem 3 *Let M_f be a connected, complete m–dimensional weighted manifold and let E be a Riemannian vector bundle of rank l over M. Denote by $\Gamma(E)$ the space of its smooth sections. Having fixed*

$$a(x) \in C^0(M), \quad A \in \mathbb{R}, \quad H \geq p$$

satisfying the further restrictions

$$p \geq A + 1, \quad p > 0,$$

let $V = V(a, f, A, p, H) \subset \Gamma(E)$ be any vector space with the following two properties.

(i) Every $\xi \in V$ has the unique continuation property, i.e., ξ is the null section whenever it vanishes on some domain.
(ii) For any $\xi \in V$, the locally Lipschitz function $u = |\xi|$ satisfies

$$\begin{cases} u(\Delta_f u + a(x)u) + A|\nabla u|^2 \geq 0 \, \text{weakly on} \, M \\ u \in L^{2p}(M_f). \end{cases}$$

If there exists a solution $0 < \varphi \in Lip_{loc}$ of the differential inequality

$$\Delta_f \varphi + Ha(x)\varphi \leq 0 \tag{8}$$

weakly outside a compact set $K \subset M$, then

$$\dim V < +\infty.$$

Proof (Outline of the proof of Theorem 3) We follow the arguments in Theorem 1.1 in [35], and we refer to it for more details. Choose $R \gg 1$ in such a way that $K \subset B_R(o)$ and, therefore, inequality (8) holds in $M \setminus B_R(o)$. Note that, by unique continuation, the restriction map

$$\begin{aligned} V &\to \Gamma(E|_{B_R}) \\ \xi &\mapsto \xi|_{B_R} \end{aligned}$$

is an injective homomorphism. Use the same symbol V to denote the image of V in $\Gamma(E|_{B_R})$. Easily adapting to the weighted setting the extension, obtained in [35, Lemma 2.1], of a classical result by P. Li we obtain that if $T \subset V$ is any finite dimensional subspace, then there exists a (non–zero) section $\bar{\xi} \in T$ such that, setting $\bar{\psi} = |\bar{\xi}|$, it holds

$$(\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M \leq \mathrm{vol}_f(B_R) \min\{l, \dim T\}^{\min(1,p)} \sup_{B_R} \bar{\psi}^{2p}. \tag{9}$$

Observe now that, on every sufficiently small closed ball,

$$\lambda_1^{L_H}(B_{3\delta}(x)) > 0,$$

where $L_H = -\Delta_f - Ha(x)$, and therefore there exists $w > 0$ solution on $B_{3\delta}(x)$ of

$$\Delta_f w + Ha(x)w = 0.$$

Let $u \geq 0$ be a locally Lipschitz, weak solution of

$$u(\Delta_f u + a(x)u) + A|\nabla u|^2 \geq 0. \tag{10}$$

Applying the computational Lemma 9 in [39] with $\beta = p \geq A + 1$, $\alpha = \frac{p}{H}$, setting $h = -\log w^{2\alpha} + f$, we deduce that

$$v = u^{\beta} w^{-\alpha}$$

satisfies

$$\Delta_h v \geq 0 \quad \text{weakly on} \, B_{3\delta}(x) \tag{11}$$

and

$$\|v\|_{L^2(M_h)} = \|u^p\|_{L^2(M_f)}\,. \tag{12}$$

Since, locally, a weighted L^2–Sobolev inequality is always available, reasoning as in Section 2 in [35], we are able to obtain the following local weighted L^1–mean value inequality for solutions v of (11)

$$\sup_{B_\delta(x)} v^2 \leq C \int_{B_{2\delta}(x)} v^2 e^{-h} d\mathrm{vol}_M$$

for some constant $C > 0$ depending on $w|_{\overline{B}_{2\delta}(x)}$ and the geometry of $B_{2\delta}(x)$. Recalling the definition of v, we deduce from the previous inequality and (12) the following weighted L^p–mean value inequality for solutions u of (8)

$$\sup_{B_\delta(x)} u^{2p} \leq C' \int_{B_{2\delta}} u^{2p} e^{-f} d\mathrm{vol}_M,$$

where

$$C' = \left(\sup_{B_\delta(x)} w^{\frac{p}{H}}\right)^2 C.$$

The local inequalities patch together and, in the special case of $\bar{\psi}$, give

$$\sup_{B_R(o)} \bar{\psi}^{2p} \leq C' \int_{B_{R+1}(o)} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M.$$

Inserting into (9) we obtain

$$\begin{aligned}(\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M &\leq C' \mathrm{vol}_f(B_R) \min\{l, \dim T\}^{\min(1,p)} \\ &\times \left(\int_{B_R} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M + \int_{A(R,R+1)} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M \right) \end{aligned} \tag{13}$$

where $A(R, R+1)$ is the annulus $B_{R+1} \setminus B_R$. Now considering a suitable combination of u and φ, adapting the proof of Lemma 2.7 in [35] to the weighted setting in a similar way to what we just did, we obtain a weighted integral, a–priori estimate on annuli of the type

$$\int_{A(R,R+1)} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M \leq C'' \int_{B_R} \bar{\psi}^{2p} e^{-f} d\mathrm{vol}_M,$$

for some constant C'' independent of $\bar{\psi}$. From this latter and (13) we finally deduce

$$\dim T \leq C''' \min\{l, \dim T\},$$

from some C''' depending only on the geometry of B_R. This proves that any finitely generated subspace T of V has dimension which is bounded by a universal constant, depending only on the rank l of E and on the weighted geometry of B_R. The same bound must work for the dimension of the whole V.

In the non–weighted case there is the well–known connection, developed by P. Li and L.–F. Tam (see e.g. [21]), between L^2 harmonic 1–forms, the number of non–parabolic ends, and the Morse index of the operator $-\Delta - a(x)$, where $-a(x)$ is the smallest eigenvalue of the Ricci tensor at x. In Theorem 4 below we shall see that an analogous relation holds in the weighted setting. This can be easily obtained with minor changes to the proofs in [21].

Recall that an end E of a weighted manifold M_f with respect to a fixed compact set D with smooth boundary is said to be f–parabolic if and only if its double is f–parabolic or, equivalently, if every positive f–superharmonic function u on E satisfying $\partial u/\partial \nu \geq 0$ on ∂E, ν being the unit outward normal to ∂E, is constant. Otherwise the end will be called non–f–parabolic. Non–f–parabolicity of the end E can be also characterized by the existence of a positive minimal Green kernel G_f for Δ_f, satisfying Neumann boundary conditions on ∂E. As we said above, the following result permits to control the number of non–f–parabolic ends by the dimension of the space of bounded f–harmonic functions with finite Dirichlet weighted integral. The idea of the proof is the same as in the non–weighted case. Given two distinct f–parabolic ends E_A and E_B, one can construct bounded f–harmonic functions g_A on M_f with finite Dirichlet weighted integral such that

$$\sup_{E_A} g_A = 1 \qquad \inf_{E_B} g_A = 0,$$

and these turn out to be linearly independent.

Theorem 4 *Let $\mathcal{H}^\infty_\mathcal{D}(M_f)$ denote the space of bounded f–harmonic functions with finite Dirichlet weighted integral on M_f, and by $N(D)$ the number of non–f–parabolic ends of M_f with respect to the relatively compact domain D. Then*

$$N(D) \leq \dim \mathcal{H}^\infty_\mathcal{D}(M_f).$$

It follows that, if $\mathcal{H}^\infty_\mathcal{D}(M_f)$ is finite dimensional, then M_f has finitely many non–f–parabolic ends, whose number is bounded above by $\dim \mathcal{H}^\infty_\mathcal{D}(M_f)$.

Let $\delta_f = \delta + i_{\nabla f}$, and denote with $\Delta^f_H = \delta_f d + d\delta_f$ the Hodge f–Laplacian on M_f. We have that the following f–Weitzenbock formula for 1–forms holds

$$\frac{1}{2}\Delta_f |\omega|^2 = -\left\langle \Delta^f_H \omega, \omega \right\rangle + |D\omega|^2 + \mathrm{Ric}_f(\omega^\sharp, \omega^\sharp).$$

In particular, if $\omega \in \mathcal{H}_1(M_f) = \left\{ 1\text{–forms}\, \omega \mid \Delta^f_H \omega = 0 \right\}$, we obtain

$$\frac{1}{2}\Delta_f |\omega|^2 = |D\omega|^2 + \mathrm{Ric}_f(\omega^\sharp, \omega^\sharp). \tag{14}$$

Thus, let $\mathrm{Ric}_f \geq -a(x)$ for some continuous function $a(x)$, and consider the vector space $L^{2,f}\mathcal{H}^1(M_f) = \left\{ \xi \in \mathcal{H}^1(M_f) \mid |\xi| \in L^2(M_f) \right\}$. Using Kato inequality, we get that, for any $\xi \in L^{2,f}\mathcal{H}^1(M_f)$, the locally Lipschitz function $u = |\xi|$ satisfies

$$\begin{cases} u(\Delta_f u + a(x) u) \geq 0 \ \text{ weakly on } M \\ \int_M u^2 e^{-f} d\mathrm{vol}_M < +\infty. \end{cases}$$

Moreover, note that equation $\Delta_H^f\omega = 0$ is equivalent to the equation $\Delta_H\omega = F(x,\omega,d\omega)$ with F satisfying the structural conditions of Aronszajn–Cordes; see e.g. Appendix A in [36]. This suffices to guarantee that every $\xi \in \mathcal{H}^1(M_f)$ has the unique continuation property.

We are thus in a situation where Theorem 3 can be applied. Hence, using Proposition 5, we obtain the following consequence of Theorem 4. Compare also with [32] where some related results are obtained.

Corollary 1 *Let M_f be a complete non–compact weighted manifold satisfying*

$$\mathrm{Ric}_f \geq -\, a(x)$$

for some nonnegative continuous function $a(x)$, and let $L = -\Delta_f - a(x)$. Suppose furthermore that L has finite Morse index. Then M_f has at most finitely many non–f–parabolic ends.

Proof (Sketch of the proof of Corollary 1) In order to apply Theorem 4 to get the conclusion in the above corollary we have used the following fact. If u is a f–harmonic function with finite Dirichlet weigthed integral, then its exterior differential du belongs to $\mathcal{H}_1(M_f)$. Moreover $du = 0$ if and only if $u \equiv const$. Hence, we have that

$$\dim\mathcal{H}^{\infty}_{\mathcal{D}}(M_f) \leq \dim\mathcal{H}_{\mathcal{D}}(M_f) \leq \dim L^{2,f}\mathcal{H}^1(M_f) + 1,$$

where we denote by $\mathcal{H}_{\mathcal{D}}(M_f)$ the space of f–harmonic functions with finite Dirichlet weighted integral on M_f.

Remark 5 As observed in [35], the generality achieved in Theorem 3 permits to deal also with situation in which we do not have the validity of a refined Kato inequality. This is essential in our case since, as observed in Remark 4.2 in [40], in general we do not have the validity of any refined Kato inequality for f–harmonic forms.

In order to deduce topological consequences from the finiteness result of the space of bounded f–harmonic functions with finite weighted Dirichlet integral on M_f, we need to find conditions which ensure that all ends of M_f are non–f–parabolic. This can be done adapting to the weighted setting a result by H. D. Cao, Y. Shen, S. Zhu, [18]. See also [5], where this result is proved in the more general setting of metric measure spaces.

Lemma 1 *Let M_f be a complete weighted manifold, and assume that for some $0 \leq \alpha < 1$, there exists a constant $S(\alpha) > 0$ such that the weighted L^2–Sobolev inequality*

$$\left(\int_M h^{\frac{2}{1-\alpha}} e^{-f} \mathrm{dvol}_M\right)^{1-\alpha} \leq S(\alpha) \int_M |\nabla h|^2 e^{-f} \mathrm{dvol}_M \tag{15}$$

holds for every smooth function compactly supported in the complement of a compact set K. Then every end E of M_f is either non–f–parabolic or it has finite f–volume.

Remark 6 Suppose that M_f supports a weighted L^1–Sobolev inequality outside a compact set K, namely for some $\alpha \in \left(1, \frac{m}{m-1}\right]$ there exists a constant $S_1(\alpha) > 0$ such that

$$\left(\int_M h^\alpha e^{-f} d\mathrm{vol}_M\right)^{\frac{1}{\alpha}} \leq S_1(\alpha) \int_M |\nabla h| e^{-f} d\mathrm{vol}_M \tag{16}$$

for every smooth function u compactly supported in $M \setminus K$. Reasoning as in the non–weighted setting, see e.g. Lemma 7.15 in [36], one can show that every end with respect to K has infinite f–volume and that, if $m \geq 3$, (15) holds with

$$S(\alpha) = \left(\frac{2S_1(\alpha)}{2-\alpha}\right)^2.$$

As a consequence of Lemma 1, it follows that if M_f supports (16) for some $\alpha \in \left(1, \frac{m}{m-1}\right]$ and for every smooth function u compactly supported in $M \setminus K$, then every end of M_f with respect to K is non–f–parabolic.

5 Weighted Hessian comparison theorem

Motivated by Remark 6, we are interested now in proving, under suitable conditions, the validity of a weighted L^1–Sobolev inequality for an hypersurface Σ isometrically immersed in a weighted manifold M_f. In the non–weighted setting, according to D. Hoffman and J. Spruck, [20], minimal submanifolds of Cartan–Hadamard manifolds enjoy an L^1–Sobolev inequality. In this order of ideas, we have to address the issue of defining a right concept of weighted sectional curvature.

In weighted geometry there are good concepts of Ricci and scalar curvature, namely, the Bakry–Émery Ricci tensor and the Perelman scalar curvature, defined on M_f as

$$P_f = R + 2\Delta f - |\nabla f|^2,$$

where R is the scalar curvature of M. On the other hand, as far as we know, there is no concept of sectional curvature associated to a weighted manifold and, in general, to a measure. As observed in [42], both Ric_f and P_f, can be viewed as the infinite–dimensional limit of their conformally invariant counterparts. Trying to carry out the same process for the full curvature tensor one easily realizes that, "letting the dimension go to infinity", the conformally invariant counterpart of the Riemann tensor recovers the Riemann tensor itself. This is not so surprising from the viewpoint of sectional curvature, since sectional curvature only takes into account two–dimensional subspaces, and hence the dimension plays no role in defining this concept. This informal discussion suggests that a good concept of sectional curvature in weighted geometry should be the sectional curvature itself. This assertion is supported by the following comparison theorem.

Theorem 5 *Let M_f^m be a complete weighted m–dimensional manifold. Having fixed a reference point $o \in M$, let $r(x) = \mathrm{dist}_M(x, o)$ and let $D_o = M \setminus \mathrm{cut}(o)$ be the domain of the normal geodesic coordinates centered at o. Given a smooth even function G on $\mathbb{R}$, let h be the solution of the Cauchy problem*

$$\begin{cases} h'' - Gh = 0 \\ h(0) = 0, \quad h'(0) = 1 \end{cases} \tag{17}$$

and let $I = [0, r_0) \subseteq [0, +\infty)$ be the maximal interval where h is positive. Suppose that the radial sectional curvature of M, that is the sectional curvature of 2–planes containing ∇r, satisfies

$$\mathrm{Sect}_{\mathrm{rad}} \geq -G(r(x)) \quad (\text{resp. } \leq) \tag{18}$$

on $B_{r_0}(o)$ and, furthermore, assume that

$$\eta(r) = \langle \nabla r, \nabla f \rangle \geq -\theta(r) \quad (\text{resp. } \leq) \tag{19}$$

for some $\theta \in C^0([0, +\infty))$, and $\eta(s) = o(1)$ as $s \to 0^+$. Let

$$\mathrm{Hess}_f(r) := \mathrm{Hess}(r) - \frac{1}{m} \langle \nabla f, \nabla r \rangle \langle \cdot\,, \cdot \rangle$$

then

$$\mathrm{Hess}_f(r) \leq \frac{h'}{h} \{\langle \cdot\,, \cdot \rangle - dr \otimes dr\} + \frac{1}{m}\theta(r) \langle \cdot\,, \cdot \rangle \quad (\text{resp. } \geq). \tag{20}$$

Remark 7 Note that tracing (20), we recover corresponding estimates for $\Delta_f r$. These are consistent with comparison results for weighted manifolds with $\mathrm{Ric}_f(\nabla r, \nabla r)$ bounded from below by $-(m-1)G(r)$ and f satisfying (19) for some non–decreasing function $\theta \in C^0([0, +\infty))$, see Theorem 3.1 in [34].

Proof Observe, first of all, that $\mathrm{Hess}(r)(\nabla r, X) = 0$ for all $X \in T_x M$ and $x \in D_o \setminus \{o\}$. Next, since $\mathrm{Hess}_f(r)$ is symmetric, $T_x M$ has an orthonormal basis consisting of eigenvectors of $\mathrm{Hess}_f(r)$. Denoting by $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$, respectively, the greatest and the smallest eigenvalues of $\mathrm{Hess}_f(r)$ in the orthogonal complement of $\nabla r(x)$, the theorem amounts to showing that on $D_o \setminus \{o\} \cap B_{r_o}(o)$

(i) if (18) and (19) hold with $\geq$, then $\lambda_{\max} \leq \frac{h'}{h}(r(x)) + \frac{1}{m}\theta(r(x))$;
(ii) if (18) and (19) hold with $\leq$, then $\lambda_{\min} \geq \frac{h'}{h}(r(x)) + \frac{1}{m}\theta(r(x))$.

Let us prove case (ii). The argument in case (i) is completely similar. Let $x \in D_o \setminus \{o\}$, and let γ be the minimizing geodesic joining o to x. We claim that $\psi = \left(\lambda_{\min} + \frac{\eta}{m}\right) \circ \gamma$ satisfies

$$\begin{cases} \psi' + \psi^2 \geq G \\ \psi(s) = \frac{1}{s} + o(1) \quad \text{as} \quad s \to 0^+ \end{cases} \tag{21}$$

Since $\phi = \frac{h'}{h}$ satisfies

$$\begin{cases} \phi' + \phi^2 = G \quad \text{on} \quad (0, r_0) \\ \phi(s) = \frac{1}{s} + o(1) \quad \text{as} \quad s \to 0^+, \end{cases} \tag{22}$$

the required conclusion follows at once from Corollary 2.2 in [36]. To prove the claim we proceed as follows. Let γ be a minimizing geodesic joining o to $\gamma(s_0) = x \in D_o \setminus \{o\}$. For every unit vector $Y \in T_x M$ such that $Y \perp \dot\gamma(s_0)$, define a vector field $Y \perp \dot\gamma$, by parallel translation along γ. Since as noted above $\mathrm{hess}(r)(\nabla r) = \nabla_{\nabla r} \nabla r{=}0$, we compute, as in [36],

$$\frac{d}{ds}(\mathrm{Hess}(r)(\gamma)(Y,Y)) + \langle \mathrm{hess}(r)(\gamma)(Y), \mathrm{hess}(r)(\gamma)(Y)\rangle = -\mathrm{Sect}_\gamma(Y \wedge \dot\gamma). \tag{23}$$

Moreover, we have that

$$\begin{aligned} \frac{d}{ds}(\mathrm{Hess}_f(r)(\gamma)(Y,Y)) &= \frac{d}{ds}(\mathrm{Hess}(r)(\gamma)(Y,Y)) - \frac{1}{m}\frac{d}{ds}\langle \nabla r \circ \gamma, \nabla f \circ \gamma\rangle \\ &= \frac{d}{ds}(\mathrm{Hess}(r)(\gamma)(Y,Y)) - \frac{1}{m}\frac{d}{ds}\eta \circ \gamma \end{aligned} \tag{24}$$

and letting

$$\mathrm{hess}_f(r)(\gamma)(Y) = \mathrm{hess}(r)(\gamma)(Y) - \frac{1}{m}(\eta \circ \gamma) Y$$

we have that

$$\begin{aligned} \langle \mathrm{hess}_f(r)(\gamma)(Y), \mathrm{hess}_f(r)(\gamma)(Y)\rangle = {} & \langle \mathrm{hess}(r)(\gamma)(Y), \mathrm{hess}(r)(\gamma)(Y)\rangle \\ & - \frac{2}{m}\mathrm{Hess}(r)(\gamma)(Y,Y)(\eta \circ \gamma) \\ & + \frac{1}{m^2}(\eta \circ \gamma)^2 . \end{aligned} \tag{25}$$

Hence, by (23), (24), (25), and the lower bound in (18), we get that along γ

$$\begin{aligned} \frac{d}{ds}(\mathrm{Hess}_f(r)(Y,Y)) + |\mathrm{hess}_f(r)(Y)|^2 \geq {} & G(r) - \frac{1}{m}\frac{d}{ds}(\eta \circ \gamma) \\ & - \frac{2}{m}\mathrm{Hess}(r)(Y,Y)(\eta \circ \gamma) \\ & + \frac{1}{m^2}(\eta \circ \gamma)^2 . \end{aligned} \tag{26}$$

Note that for any unit vector field $X \perp \nabla r$

$$\mathrm{Hess}_f(r)(\gamma)(X,X) \geq \lambda_{\mathrm{min.}}$$

Thus, if Y is choosen so that, at s_0

$$\mathrm{Hess}_f(r)(\gamma)(Y,Y) = \lambda_{\min}(\gamma(s_0)),$$

then the function $\mathrm{Hess}_f(r)(\gamma)(Y,Y) - \lambda_{\min} \circ \gamma$ attains its minimum at $s = s_0$ and, if at this point $\lambda_{\min}$ is differentiable, then its derivative vanishes:

$$\left.\frac{d}{ds}\right|_{s_0} \mathrm{Hess}_f(r)(\gamma)(Y,Y) - \left.\frac{d}{ds}\right|_{s_0} \lambda_{\min} \circ \gamma = 0.$$

Whence, using (26), we obtain that, at s_0,

$$\begin{aligned}
\frac{d}{ds}(\lambda_{\min} \circ \gamma) + (\lambda_{\min} \circ \gamma)^2 &\geq G(r) - \frac{1}{m}\frac{d}{ds}(\eta \circ \gamma) \\
&\quad - \frac{2}{m}\mathrm{Hess}(r)(Y,Y)\,(\eta \circ \gamma) + \frac{1}{m^2}(\eta \circ \gamma)^2 \\
&= G(r) - \frac{1}{m}\frac{d}{ds}(\eta \circ \gamma) \\
&\quad - \frac{2}{m}\mathrm{Hess}_f(r)(Y,Y)\,(\eta \circ \gamma) - \frac{1}{m^2}(\eta \circ \gamma)^2 \\
&= G(r) - \frac{d}{ds}\frac{\eta \circ \gamma}{m} - 2(\lambda_{\min} \circ \gamma)\frac{\eta \circ \gamma}{m} - \frac{(\eta \circ \gamma)^2}{m^2}.
\end{aligned}$$

Letting now $\psi = \left(\lambda_{\min} + \frac{\eta}{m}\right) \circ \gamma$ we get the desired differential inequality (21). The asymptotic behaviour

$$\psi(s) = \frac{1}{s} + o(1) \quad \text{as} \quad s \to 0^+$$

follows from our assumptions on η and the fact that

$$\mathrm{Hess}(r) = \frac{1}{r}(\langle \cdot\,, \cdot \rangle - dr \otimes dr) + o(1) \quad \text{as} \quad r \to 0^+.$$

6 A Sobolev inequality in the weighted setting

In this section we prove a general weighted L^1–Sobolev inequality for submanifolds Σ^m of a weighted manifold M_f^{m+1}, satisfying some restrictions on f and on the sectional curvature of M. The proof is inspired by the papers of J. H Michael, and L. M. Simon, [27], and of D. Hoffman and J. Spruck, [20]. Recall that with $\overline{(\cdot)}$ we refer to quantities in the ambient space.

Theorem 6 *Let $\Sigma^m \to M_f^{m+1}$ be an isometric immersion. Assume that $\overline{\mathrm{Sect}} \leq 0$ and suppose that there exists a positive constant c_m such that*

$$\limsup_{\rho \to 0^+} \frac{\mathrm{vol}_f(S_\rho(\xi))}{\rho^m} \geq c_m, \tag{27}$$

for almost all $\xi \in M_f$, where we are using the notation

$$S_\rho(\xi) = \left\{ x \in \Sigma \,|\, {}^M\mathrm{dist}(x,\xi) \leq \rho \right\}.$$

Let h be a non–negative compactly supported C^1 function on Σ. Then

$$\left[\int_\Sigma h^{\frac{m}{m-1}} e^{-f} d\mathrm{vol}_\Sigma\right]^{\frac{m-1}{m}} \leq C\left[\int_\Sigma |\nabla h| + h\left(|H_f| + |\overline{\nabla} f|\right) e^{-f} d\mathrm{vol}_\Sigma\right]. \tag{28}$$

Remark 8 Note that for every isometric immersion $\Sigma^m \to M^{m+1}$ we have that

$$\limsup_{\rho\to 0^+} \frac{\mathrm{vol}(S_\rho(\xi))}{\rho^m} \geq \omega_m$$

for almost all $\xi \in M$, with ω_m the volume of the unit ball in $\mathbb{R}^m$. Hence condition (27) is satisfied if we assume that $f < f^* < +\infty$.

Remark 9 Theorem 6 has a companion weighted isoperimetric inequality. In this regard, we mention that the isoperimetric problem in Riemannian manifolds with density (and in particular in the Gaussian space) is a recent and very active field of research; see e.g. [29], [41], [30], [28].

Let $\Sigma^m \to M_f^{m+1}$ be an isometric immersion as in Theorem 6, let $X = r\overline{\nabla} r$, r distance function on M, and $\{E_1, \ldots, E_m\}$ be a local orthonormal frame on Σ. Since

$$^{\Sigma}\mathrm{div}X = \sum_{i=1}^{m} \left\langle \overline{\nabla}_{E_i}(r\overline{\nabla} r), E_i \right\rangle = r\sum_{i=1}^{m} \overline{\mathrm{Hess}}(r)(E_i, E_i) + \sum_{i=1}^{m} \left\langle \overline{\nabla} r, E_i \right\rangle^2$$

we obtain, by the classical Hessian comparison theorem, that

$$\begin{aligned} ^{\Sigma}\mathrm{div}X - \left\langle \overline{\nabla} f, X \right\rangle &= r\sum_{i=1}^{m} \overline{\mathrm{Hess}}(r)(E_i, E_i) + \sum_{i=1}^{m} \left\langle \overline{\nabla} r, E_i \right\rangle^2 \\ &\quad - r\left\langle \overline{\nabla} f, \overline{\nabla} r \right\rangle \\ &\geq m - \sum_{i=1}^{m} \left\langle \overline{\nabla} r, E_i \right\rangle^2 + \sum_{i=1}^{m} \left\langle \overline{\nabla} r, E_i \right\rangle^2 - r\left\langle \overline{\nabla} f, \overline{\nabla} r \right\rangle \\ &\geq m - r|\overline{\nabla} f|. \end{aligned} \tag{29}$$

Let λ be a non–decreasing C^1 function on $\mathbb{R}$ with $\lambda(t) = 0$ for $t \leq 0$. Let $0 \leq h \in C_c^1(\Sigma)$. For $\xi \in \Sigma$, let $r(x)$ be the distance function from the point ξ on M. Then we define the following quantities

$$\begin{aligned} \phi_\xi(\rho) &= \int_\Sigma \lambda(\rho - r(x))h(x)e^{-f}d\mathrm{vol}_\Sigma; \\ \psi_\xi(\rho) &= \int_\Sigma \lambda(\rho - r(x))\left(|\nabla h(x)| + h(x)|H_f(x)|\right)e^{-f}d\mathrm{vol}_\Sigma; \\ \mu_\xi(\rho) &= \int_\Sigma \lambda(\rho - r(x))\left(|\overline{\nabla} f|(x)\, h(x)\right)e^{-f}d\mathrm{vol}_\Sigma; \\ \overline{\phi}_\xi(\rho) &= \int_{S_\rho(\xi)} h(x)e^{-f}d\mathrm{vol}_\Sigma; \\ \overline{\psi}_\xi(\rho) &= \int_{S_\rho(\xi)} \left(|\nabla h(x)| + h(x)|H_f(x)|\right)e^{-f}d\mathrm{vol}_\Sigma; \\ \overline{\mu}_\xi(\rho) &= \int_{S_\rho(\xi)} \left(|\overline{\nabla} f|(x)h(x)\right)e^{-f}d\mathrm{vol}_\Sigma; \end{aligned}$$

We now prove two lemmas which generalize Lemmas 4.1 and 4.2 in [20]. The first one relates the growth of $\phi_\xi(\rho)$ to $\psi_\xi(\rho)$ and $\mu_\xi(\rho)$.

Lemma 2 *Let $\Sigma^m \to M_f^{m+1}$ be an isometric immersion. Assume that $\overline{\mathrm{Sect}} \leq 0$. Then*

$$-\frac{d}{d\rho}\left(\rho^{-m}\phi_\xi(\rho)\right) \leq \rho^{-m}\left[\psi_\xi(\rho)+\mu_\xi(\rho)\right]. \tag{30}$$

Proof Let X be the radial vector field centered at ξ and let $(\cdot)^T$ denote the projection on the tangent bundle of Σ. Since

$$\begin{aligned}{}^{\Sigma}\mathrm{div}_f(\lambda(\rho-r)hX^T) &= \lambda(\rho-r)h\,{}^{\Sigma}\mathrm{div}_f(X^T) - \lambda'(\rho-r)h\left\langle X^T,\nabla r\right\rangle \\ &\quad +\lambda(\rho-r)\left\langle \nabla h, X^T\right\rangle,\end{aligned}$$

and

$$\begin{aligned}{}^{\Sigma}\mathrm{div}_f(X^T) &= \sum_{i=1}^m\left\langle \overline{\nabla}_{E_i}X^T,E_i\right\rangle - \left\langle \nabla f, X^T\right\rangle \\ &= \sum_{i=1}^m\left\langle \overline{\nabla}_{E_i}X,E_i\right\rangle - \langle X,\nu\rangle\sum_{i=1}^m\left\langle \overline{\nabla}_{E_i}\nu,E_i\right\rangle \\ &\quad -\left\langle \overline{\nabla}f,X\right\rangle + \left\langle \overline{\nabla}f,\nu\right\rangle\langle X,\nu\rangle \\ &= {}^{\Sigma}\mathrm{div}X - \left\langle \overline{\nabla}f,X\right\rangle + \langle X,\nu\rangle\left(H+\left\langle \overline{\nabla}f,\nu\right\rangle\right) \\ &= {}^{\Sigma}\mathrm{div}X - \left\langle \overline{\nabla}f,X\right\rangle + \langle X,\nu\rangle H_f,\end{aligned}$$

we obtain that

$$\begin{aligned}{}^{\Sigma}\mathrm{div}_f\left(\lambda(\rho-r)hX^T\right) &= \lambda(\rho-r)h\left({}^{\Sigma}\mathrm{div}X - \left\langle \overline{\nabla}f,X\right\rangle\right) \\ &\quad +\lambda(\rho-r)h\langle X,\nu\rangle H_f - \lambda'(\rho-r)h\left\langle X^T,\nabla r\right\rangle \\ &\quad +\lambda(\rho-r)\left\langle \nabla h, X^T\right\rangle.\end{aligned} \tag{31}$$

Since $|\nabla r| = \left|(\overline{\nabla}r)^T\right| \leq |\overline{\nabla}r| = 1$ and $\lambda(\rho-r)=\lambda'(\rho-r)=0$ for $r\geq\rho$, integrating (31) over Σ with respect to the weighted volume measure and using the f–divergence theorem, we get that

$$\begin{aligned}\int_\Sigma \lambda(\rho-r)h\left({}^{\Sigma}\mathrm{div}X - \left\langle X,\overline{\nabla}f\right\rangle\right)e^{-f}d\mathrm{vol}_\Sigma &= \int_\Sigma \lambda'(\rho-r)h\left\langle X^T,\nabla r\right\rangle e^{-f}d\mathrm{vol}_\Sigma \\ &\quad -\int_\Sigma \lambda(\rho-r)h\,H_f\,r\left\langle \overline{\nabla}r,\nu\right\rangle e^{-f}d\mathrm{vol}_\Sigma \\ &\quad -\int_\Sigma \lambda(\rho-r)r\langle\nabla r,\nabla h\rangle e^{-f}d\mathrm{vol}_\Sigma \\ &\leq \int_\Sigma r\lambda'(\rho-r)|h|e^{-f}d\mathrm{vol}_\Sigma \\ &\quad +\int_\Sigma r\,\lambda(\rho-r)|h|\,|H_f|e^{-f}d\mathrm{vol}_\Sigma \\ &\quad +\int_\Sigma r\lambda(\rho-r)|\nabla h|e^{-f}d\mathrm{vol}_\Sigma \\ &\leq \rho\phi'_\xi(\rho)+\rho\psi_\xi(\rho).\end{aligned}$$

Hence, by (29) we have that

$$\begin{aligned}\rho\phi_\xi'(\rho)+\rho\psi_\xi(\rho) &\geq \int_\Sigma (m-r|\overline{\nabla}f|)\lambda(\rho-r)he^{-f}\mathrm{dvol}_\Sigma \\ &= m\phi_\xi(\rho)-\int_\Sigma \lambda(\rho-r)r|\overline{\nabla}f|he^{-f}\mathrm{dvol}_\Sigma \\ &\geq m\phi_\xi(\rho)-\rho\int_\Sigma \lambda(\rho-r)|\overline{\nabla}f|he^{-f}\mathrm{dvol}_\Sigma,\end{aligned}$$

that is,

$$m\phi_\xi(\rho)-\rho\mu_\xi(\rho)\leq \rho\phi_\xi'(\rho)+\rho\psi_\xi(\rho), \tag{32}$$

proving (30).

Lemma 3 *Let $\xi\in\Sigma$ be such that $h(\xi)\geq 1$. Let α,t satisfy $0<\alpha<1\leq t$, and suppose that there exists a constant c_m such that (27) holds. Set*

$$\rho_0=\frac{1}{1-\alpha}\left[c_m^{-1}\int_\Sigma he^{-f}\mathrm{dvol}_\Sigma\right]^{\frac{1}{m}}.$$

Then there exists ρ, $0<\rho<\rho_0$, such that

$$\overline{\phi}_\xi(t\rho)\leq \alpha^{-1}t^{m-1}\rho_0\left[\overline{\psi}_\xi(\rho)+\overline{\mu}_\xi(\rho)\right].$$

Proof Integrating (30) on (σ,ρ_0), $\sigma\in(0,\rho_0)$, we have that

$$\sigma^{-m}\phi_\xi(\sigma)\leq \rho_0^{-m}\phi_\xi(\rho_0)+\int_0^{\rho_0}\rho^{-m}\psi_\xi(\rho)d\rho+\int_0^{\rho_0}\rho^{-m}\mu_\xi(\rho)d\rho.$$

Take $\varepsilon\in(0,\sigma)$ and choose λ such that $\lambda(t)=1$ for $t\geq\epsilon$. Then

$$\sigma^{-m}\overline{\phi}_\xi(\sigma-\epsilon)\leq \rho_0^{-m}\overline{\phi}_\xi(\rho_0)+\int_0^{\rho_0}\rho^{-m}\overline{\psi}_\xi(\rho)d\rho+\int_0^{\rho_0}\rho^{-m}\overline{\mu}_\xi(\rho)d\rho.$$

Hence, since σ,ε are arbitrary,

$$\sup_{\sigma\in(0,\rho_0)}\sigma^{-m}\overline{\phi}_\xi(\sigma)\leq \rho_0^{-m}\overline{\phi}_\xi(\rho_0)+\int_0^{\rho_0}\rho^{-m}\overline{\psi}_\xi(\rho)d\rho+\int_0^{\rho_0}\rho^{-m}\overline{\mu}_\xi(\rho)d\rho.$$

By contradiction, assume that for all $\rho\in(0,\rho_0)$,

$$\overline{\psi}_\xi(\rho)+\overline{\mu}_\xi(\rho)<\alpha t^{-(m-1)}\rho_0^{-1}\overline{\phi}_\xi(t\rho).$$

Then

$$\begin{aligned}&\int_0^{\rho_0}\rho^{-m}\overline{\psi}_\xi(\rho)d\rho+\int_0^{\rho_0}\rho^{-m}\overline{\mu}_\xi(\rho)d\rho \\ &\qquad< \alpha\rho_0^{-1}\int_0^{\rho_0}t^{-(m-1)}\overline{\phi}_\xi(t\rho)\rho^{-m}d\rho \\ &\qquad= \alpha\rho_0^{-1}\int_0^{t\rho_0}s^{-m}\overline{\phi}_\xi(s)ds \\ &\qquad\leq \alpha\rho_0^{-1}\left[\int_0^{\rho_0}s^{-m}\overline{\phi}_\xi(s)ds+\int_{\rho_0}^{+\infty}s^{-m}\overline{\phi}_\xi(s)ds\right] \\ &\qquad\leq \alpha\sup_{\sigma\in(0,\rho_0)}\sigma^{-m}\overline{\phi}_\xi(\sigma)+\alpha\rho_0^{-m}(m-1)^{-1}\int_\Sigma he^{-f}\mathrm{dvol}_\Sigma.\end{aligned}$$

Thus we get that

$$(1-\alpha)\sup_{\sigma\in(0,\rho_0)}\sigma^{-m}\overline{\phi}_\xi(\sigma)<\rho_0^{-m}\int_\Sigma he^{-f}d\mathrm{vol}_\Sigma\left[1+\alpha(m-1)^{-1}\right].$$

Using (27), this gives a contradiction.

Proof (Proof of Theorem 6) We follow the argument in [27], [20].
Let $A=\{\xi\in\Sigma\,|\,h(\xi)\geq 1\}$. Set $\rho_i=\beta^i\rho_0$, where $\frac{2}{t}<\beta<1$, $t>2$. Define

$$A_i=\left\{\xi\in A\,|\,\overline{\phi}_\xi(t\rho)\leq\alpha^{-1}t^{m-1}\rho_0\left[\overline{\psi}_\xi(\rho)+\overline{\mu}_\xi(\rho)\right]\text{ for some }\rho\in(\rho_{i+1},\rho_i)\right\}.$$

It follows from Lemma 3 that $A=\bigcup_{i=0}^\infty A_i$. Next, define inductively a sequence $F_0,F_1,\ldots$ of subsets of A as follows:

1. $F_0=\emptyset$;
2. Let $k\geq 1$ and assume $F_0,F_1,\ldots,F_{k-1}$ have been defined. Let $B_k=A_k\setminus\bigcup_{i=0}^{k-1}\bigcup_{\xi\in F_i}S_{\beta t\rho_i}(\xi)$.

If $B_k=\emptyset$, then put $F_k=\emptyset$. If $B_k\neq\emptyset$, define F_k to be a finite subset of B_k such that $B_k\subset\bigcup_{\xi\in F_k}S_{\beta t\rho_k}(\xi)$ and the sets $S_{\rho_k}(\xi)$ are pairwise disjoint. Then one checks that the following properties hold:

(a) $F_i\subset A_i$;
(b) $A\subset\bigcup_{i=1}^\infty\bigcup_{\xi\in F_i}S_{\beta t\rho_i}(\xi)$;
(c) For all i, $\{S_{\rho_i}(\xi)\}_{\xi\in F_i}$ is a collection of pairwise disjoint sets.

Let $\xi\in F_i$. Then, by property (a) we have that, for some $\rho\in(\beta\rho_i,\rho_i)$,

$$\overline{\phi}_\xi(t\rho)\leq\alpha^{-1}t^{m-1}\rho_0\left[\overline{\psi}_\xi(\rho)+\overline{\mu}_\xi(\rho)\right].$$

Thus, since $\theta\leq 0$, $\overline{\mu}_\xi(\rho)$ is non–decreasing and hence

$$\begin{aligned}\overline{\phi}_\xi(\beta t\rho_i)&\leq\overline{\phi}_\xi(t\rho)\leq\alpha^{-1}t^{m-1}\rho_0\left[\overline{\psi}_\xi(\rho)+\overline{\mu}_\xi(\rho)\right]\\&\leq\alpha^{-1}t^{m-1}\rho_0\left[\overline{\psi}_\xi(\rho_i)+\overline{\mu}_\xi(\rho_i)\right].\end{aligned}$$

Summing over all $\xi\in F_i$ and i and using properties (b) and (c) defining $\Sigma_s=\{\xi\in\Sigma\,|\,h(\xi)\geq s\}$, we get that

$$\begin{aligned}\mathrm{vol}_f(\Sigma_1)&=\sum_{i=1}^\infty\sum_{\xi\in F_i}\mathrm{vol}_f\left(S_{\beta t\rho_i}(\xi)\cap\Sigma\right)\leq\sum_{i=1}^\infty\sum_{\xi\in F_i}\overline{\phi}_\xi(\beta t\rho_i)\\&\leq\sum_{i=1}^\infty\sum_{\xi\in F_i}\left[\alpha^{-1}t^{m-1}\rho_0\left(\overline{\psi}_\xi(\rho_i)+\overline{\mu}_\xi(\rho_i))\right)\right]\\&\leq\alpha^{-1}t^{m-1}\rho_0\left[\int_\Sigma\left(|\nabla h|+h|H_f|\right)e^{-f}d\mathrm{vol}_\Sigma+\int_\Sigma|\overline{\nabla}f|he^{-f}d\mathrm{vol}_\Sigma\right].\end{aligned}$$

Now let $s,\varepsilon>0$ be arbitrary and let $\lambda\in C^1(\mathbb{R})$ be non–decreasing and such that $\lambda(t)=0$ for $t\leq-\varepsilon$ and $\lambda(t)=1$ for $t\geq 0$. Since we have also that

$$\Sigma_s=\{\xi\in\Sigma\,|\,\lambda(h(x)-s)\geq 1\},$$

replacing h by $\lambda(h-s)$ in the last computation, one obtains

$$\begin{aligned}\mathrm{vol}_f(\Sigma_s) \le \frac{\alpha^{-1}}{1-\alpha} t^{m-1} \left[c_m^{-1} \int_\Sigma \lambda(h-s) e^{-f} d\mathrm{vol}_\Sigma \right]^{\frac{1}{m}} \\ \times \left[\int_\Sigma \lambda'(h-s)|\nabla h| + \lambda(h-s) \left[|H_f| + |\overline{\nabla} f| \right] e^{-f} d\mathrm{vol}_\Sigma \right].\end{aligned} \tag{33}$$

Multiplying both sides of (33) by $s^{\frac{1}{m-1}}$, using the fact that $\lambda(h-s)=0$ for $s \ge h+\varepsilon$, and letting $c = \alpha^{-1}(1-\alpha)^{-1} t^{m-1} c_m^{-\frac{1}{m}}$, we obtain

$$\begin{aligned}s^{\frac{1}{m-1}} \mathrm{vol}_f(\Sigma_s) \le c \left[\int_\Sigma (h+\varepsilon)^{\frac{m}{m-1}} e^{-f} d\mathrm{vol}_\Sigma \right]^{\frac{1}{m}} \\ \times \left[\int_\Sigma \lambda'(h-s)|\nabla h| + \lambda(h-s) \left(|H_f| + |\overline{\nabla} f| \right) e^{-f} d\mathrm{vol}_\Sigma \right].\end{aligned}$$

Finally, we integrate over $(0,+\infty)$ with respect to s and let $\varepsilon \to 0$. The desired inequality (28) follows noting that

$$\begin{aligned}\int_0^{+\infty} s^{\frac{1}{m-1}} \mathrm{vol}_f(\Sigma_s) dt &= \int_0^{+\infty} s^{\frac{1}{m-1}} \left(\int_{\Sigma_s} e^{-f} d\mathrm{vol}_\Sigma \right) ds \\ &= \frac{m-1}{m} \int_\Sigma \left[\int_0^h \frac{m}{m-1} s^{\frac{1}{m-1}} ds \right] e^{-f} d\mathrm{vol}_\Sigma \\ &= \frac{m-1}{m} \int_\Sigma h^{\frac{m}{m-1}} e^{-f} d\mathrm{vol}_\Sigma, \\ \int_0^{+\infty} \lambda(h-s) ds &\le h+\varepsilon, \\ \int_0^{+\infty} \lambda'(h-s) ds &\le 1.\end{aligned}$$

7 Topological results

By Gauss equation it is not difficult to see that, given an f–minimal hypersurface $x : \Sigma^m \to M_f^{m+1}$, the Bakry–Émery Ricci tensor of Σ satisfies

$$\mathrm{Ric}_f(X,X) = \overline{\mathrm{Ric}}_f(X,X) - \overline{\mathrm{Sect}}(X,\nu)|X|^2 - \langle A^2 X, X \rangle, \tag{34}$$

for any $X \in T\Sigma$. Assume now that $\overline{\mathrm{Sect}} \le 0$ and $\overline{\mathrm{Ric}}_f \ge k$. Then

$$\mathrm{Ric}_f \ge k - |\mathbf{A}|^2, \tag{35}$$

and, if $Ind_f(\Sigma) < +\infty$ and $k \ge 0$, we obtain that there exists a solution $\varphi > 0$ of the differential inequality

$$\Delta_f \varphi + a(x)\varphi \le 0,$$

weakly outside a compact set, where $a(x) = |\mathbf{A}|^2 - k$. Hence the assumptions in Corollary 1 are met and we can conclude that Σ has at most finitely many non–f–parabolic ends. Applying Theorem 6, we can now get the following

Theorem 7 *Let Σ^m be a complete f–minimal hypersurface isometrically immersed with $Ind_f(\Sigma) < +\infty$ in a complete weighted manifold M_f^{m+1} with $\overline{\mathrm{Sect}} \le 0$ and $\overline{\mathrm{Ric}}_f \ge k \ge 0$. Suppose furthermore that $f \le f^* < +\infty$ and $|\overline{\nabla} f| \in L^m(\Sigma_f)$. Then Σ has finitely many ends.*

Proof By Theorem 6 and using the f–minimality, we have that for every $0 \le h \in C_c^\infty(\Sigma)$

$$\left[\int_\Sigma h^{\frac{m}{m-1}} e^{-f} d\mathrm{vol}_\Sigma\right]^{\frac{m-1}{m}} \le C \left[\int_\Sigma |\nabla h| + h|\overline{\nabla} f| e^{-f} d\mathrm{vol}_\Sigma\right].$$

Since we are assuming that $|\overline{\nabla} f| \in L^m(\Sigma_f)$, for a suitable compact K we can suppose that

$$\|\overline{\nabla} f\|_{L^m(\Sigma\setminus K,\, e^{-f} d\mathrm{vol}_\Sigma)} < C^{-1}.$$

Then, applying the Hölder inequality, the term involving θ can be absorbed in the left–hand side, showing that the L^1–Sobolev inequality

$$\left[\int_\Sigma h^{\frac{m}{m-1}} e^{-f} d\mathrm{vol}_\Sigma\right]^{\frac{m-1}{m}} \le D \left[\int_\Sigma |\nabla h| e^{-f} d\mathrm{vol}_\Sigma\right]$$

holds for every smooth non-negative function compactly supported in $\Sigma \setminus K$ and some constant $D > 0$. By Remark 6 we hence conclude the proof.

In the discussion just above Theorem 7 we needed the hypothesis on $Ind_f(\Sigma)$, jointly with $\overline{\mathrm{Ric}}_f \ge k \ge 0$ in order to guarantee the finiteness of the Morse index of the operator $-\Delta_f - (|\mathbf{A}|^2 - k)$. Note that, on the other hand, in case $k \ge 0$ we have even that $\mathrm{Ric}_f \ge -|\mathbf{A}|^2$. To apply Corollary 1 it thus suffices to guarantee the finiteness of the Morse index of the operator $L_{\mathbf{A}} = -\Delta_f - |\mathbf{A}|^2$. In particular, adapting ideas in [24], we are going to show that this can be done assuming the finiteness of weighted total curvature.

Slightly adapting the proof in [24], it is easy to obtain the following weighted version of Theorem 2 in [24].

Lemma 4 *Let Σ^m, $m \ge 3$, be a complete non–compact Riemannian manifold enjoing the L^2–weighted Sobolev inequality*

$$\left(\int_\Sigma h^{\frac{2m}{m-2}} e^{-f} d\mathrm{vol}_\Sigma\right)^{\frac{m-2}{m}} \le C(m) \left(\int_\Sigma |\nabla h|^2 e^{-f} d\mathrm{vol}_\Sigma\right) \quad \forall\, h \in C_c^\infty(\Sigma). \tag{36}$$

Let $D \subseteq \Sigma$ be a bounded domain. Suppose $q(x)$ is a positive function defined on D and let μ_k be the k^{th} eigenvalue for

$$\begin{cases} \Delta_f \psi(x) = -\mu q(x)\psi(x) & \text{on } D \\ \psi|_{\partial D} \equiv 0 \end{cases}$$

Then

$$\mu_k^{\frac{m}{2}} \int_D q^{\frac{m}{2}} e^{-f} d\mathrm{vol}_\Sigma \geq k\tilde{C}(m).$$

Using the same idea as in [24], we can prove the following

Proposition 8 *Let $\Sigma^m \to M_f^{m+1}$, $m \geq 3$, be a complete isometrically immersed hypersurface enjoying the L^2–weighted Sobolev inequality* (36)*. Set $L_{\mathbf{A}} = -\Delta_f - |\mathbf{A}|^2$. Then*

$$Ind^{L_{\mathbf{A}}}(\Sigma) \leq \tilde{C}(m) \int_\sigma |\mathbf{A}|^m e^{-f} d\mathrm{vol}_\Sigma.$$

Proof Up to taking an exhaustion of Σ by compact domains $\{\Omega_i\}_{i=1}^\infty$, it suffices to show that

$$Ind^{L_{\mathbf{A}}}(\Omega) \leq \tilde{C}(m) \int_\Omega |\mathbf{A}|^m e^{-f} d\mathrm{vol}_\Sigma$$

for any given domain $\Omega \subseteq \Sigma$. On the other hand, consider the eigenvalue problem

$$\begin{cases} \Delta_f \psi = -\mu |\mathbf{A}|^2 \psi \quad \text{on} \quad \Omega \\ \psi|_{\partial\Omega} \equiv 0. \end{cases} \tag{37}$$

It is not difficult to prove that

$$Ind^{L_{\mathbf{A}}}(\Omega) = \sharp \left\{ \mu_k \leq 1 \mid \mu_k \text{ is an eigenvalue of (37)} \right\}. \tag{38}$$

Indeed this follows from the identity

$$\frac{\int \left(|\nabla \psi|^2 - |\mathbf{A}|^2 \psi^2 \right) e^{-f} d\mathrm{vol}_\Sigma}{\int \psi^2 e^{-f} d\mathrm{vol}_\Sigma} = \frac{\int |\mathbf{A}|^2 \psi^2 e^{-f} d\mathrm{vol}_\Sigma}{\int \psi^2 e^{-f} d\mathrm{vol}_\Sigma} \left[\frac{\int |\nabla \psi|^2 e^{-f} d\mathrm{vol}_\Sigma}{\int |\mathbf{A}|^2 \psi^2 e^{-f} d\mathrm{vol}_\Sigma} - 1 \right],$$

observing that

$$\frac{\int |\nabla \psi|^2 e^{-f} d\mathrm{vol}_\Sigma}{\int |\mathbf{A}|^2 \psi^2 e^{-f} d\mathrm{vol}_\Sigma}$$

is the quadratic form associated to the operator $-\frac{\Delta_f}{|\mathbf{A}|^2}$. Hence, if μ_k is the greatest eigenvalue of (37) less then or equal to 1, it follows by Lemma 4 that

$$Ind^{L_{\mathbf{A}}} = k \leq \tilde{C}(m) \mu_k^{\frac{m}{2}} \int_\Omega |\mathbf{A}|^m e^{-f} d\mathrm{vol}_\Sigma \leq \tilde{C}(m) \int_\Omega |\mathbf{A}|^m e^{-f} d\mathrm{vol}_\Sigma.$$

As a consequence of Proposition 8 we can now state the announced corollary of Theorem 7.

Corollary 2 *Let Σ^m be a complete f–minimal hypersurface isometrically immersed in a complete weighted manifold M_f^{m+1} with $\overline{\mathrm{Sect}} \leq 0$ and $\overline{\mathrm{Ric}}_f \geq k \geq 0$. Assume that $|\mathbf{A}| \in L^m(\Sigma_f)$. Suppose furthermore that $f \leq f^* < +\infty$ and $|\overline{\nabla} f| \in L^m(\Sigma_f)$. Then Σ has finitely many ends.*

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